Publ. Math. Debrecen
103/1-2 (2023), 165–170
DOI: 10.5486/PMD.2023.9446

Finite groups with few non-isolated subgroups

By MARIUS TĂRNĂUCEANU (Iaşi)

Abstract. Given a finite group G, we denote by $L(G)$ the subgroup lattice of G, and by $\text{Isolated}(G)$ the set of isolated subgroups of G. In this note, we describe finite groups G such that $|\text{Isolated}(G)| = |L(G)| - k$, where $k = 0, 1, 2$.

1. Introduction

Let G be a finite group, and $L(G)$ be the subgroup lattice of G. We say that a subgroup $H \in L(G)$ is *isolated* in G if for every $x \in G$, we have either $x \in H$ or $\langle x\rangle \cap H = 1$. Groups with isolated subgroups were studied in [1] and [2]. However, this concept appears much earlier (see, for instance, Section 66 of [7], and the entry "isolated subgroups" in the Encyclopedia of Mathematics [4]).

Recently, the set $\text{Isolated}(G)$ of isolated subgroups of G has been investigated for certain classes of non-abelian p-groups [6], and for abelian groups [11]. Also, the structure of finite p-groups with isolated minimal non-abelian subgroups and of finite p-groups with an isolated metacyclic subgroup has been determined in [9].

The current note deals with the following natural question:

How large can the set $\text{Isolated}(G)$ *be?*

Our main result is stated as follows. We recall that a group is called a CP_1-*group* if all its elements have prime order; the structure of such a group is given in [5] (see also [3]).

Mathematics Subject Classification: Primary: 20D30; Secondary: 20D60.
Key words and phrases: finite group, subgroup lattice, isolated subgroup, CP_1-group.

Theorem 1.1. *Let G be a finite group. Then*

(a) $\text{Isolated}(G) = L(G)$ *if and only if* G *is a* CP_1*-group.*

(b) $|\text{Isolated}(G)| = |L(G)| - 1$ *if and only if* $G \cong \mathbb{Z}_{p^2}$ *with* p *a prime.*

(c) $|\text{Isolated}(G)| = |L(G)| - 2$ *if and only if* $G \cong \mathbb{Z}_{p^3}$ *with* p *a prime or* $G \cong \mathbb{Z}_{pq}$ *with* p *and* q *distinct primes.*

The classification of finite groups G such that

$$|\text{Isolated}(G)| = |L(G)| - k$$

can be continued for $k \geq 3$. We remark that in a cyclic group G, every proper subgroup is non-isolated, so G has exactly k non-isolated subgroups if $|G|$ has exactly k proper divisors. This shows that for every positive integer k, there is a finite group with exactly k non-isolated subgroups.

For the proof of the above theorem, we need the following well-known results:

- A finite group with one or two maximal subgroups is cyclic.
- A finite group has a unique minimal subgroup if and only if it is either a cyclic p-group or a generalized quaternion 2-group.
- A finite group has exactly two minimal subgroups if and only if it is either cyclic of order $p^m q^n$ with p, q distinct primes or a direct product of a cyclic p-group of odd order and a generalized quaternion 2-group.

By a *generalized quaternion* 2*-group*, we mean a group of order 2^n for some positive integer $n \geq 3$, defined by

$$Q_{2^n} = \langle a, b \mid a^{2^{n-2}} = b^2, a^{2^{n-1}} = 1, b^{-1}ab = a^{-1} \rangle.$$

Finally, we note that the dual problem of finding finite groups with few isolated subgroups seems to be also very interesting. We say that a finite group is *isolated-simple* if it has no proper isolated subgroups. Some obvious examples of such groups are cyclic groups, generalized quaternion 2-groups or abelian p-groups of type $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k}}$ with $2 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$ (see [11, Lemma 2.3]).

Open problem. Determine the structure of isolated-simple groups.

Most of our notation is standard and will not be repeated here. Basic definitions and results on group theory can be found in [7] and [10]. For subgroup lattice concepts, we refer the reader to [8].

2. Proof of Theorem 1.1

(a) Assume that Isolated$(G) = L(G)$, but G is not a CP$_1$-group. Then there exists $a \in G$ with $o(a) = pq$, where p and q are possibly equal primes. Since $a \notin \langle a^p \rangle$ and $\langle a \rangle \cap \langle a^p \rangle = \langle a^p \rangle \neq 1$, it follows that $\langle a^p \rangle$ is not isolated, a contradiction.

Conversely, let $H \in L(G)$ and $a \notin H$. Then, $\langle a \rangle \cap H \leq \langle a \rangle$ and $\langle a \rangle \cap H \neq \langle a \rangle$. Since $\langle a \rangle$ is of prime order, we get $\langle a \rangle \cap H = 1$. This shows that $H \in$ Isolated(G), as desired.

(b) Assume that Isolated$(G) = L(G) \setminus \{H_0\}$. Then all conjugates of H_0 are also not isolated, and therefore H_0 is normal in G.

Let $a \in G \setminus H_0$ such that $\langle a \rangle \cap H_0 \neq 1$. Then, $a \notin \langle a \rangle \cap H_0$ and

$$\langle a \rangle \cap (\langle a \rangle \cap H_0) = \langle a \rangle \cap H_0 \neq 1,$$

which implies that $\langle a \rangle \cap H_0$ is not isolated in G. It follows that $\langle a \rangle \cap H_0 = H_0$, i.e., $H_0 \subseteq \langle a \rangle$. Clearly, H_0 cannot have proper subgroups because they would not be isolated, contradicting our assumption. Thus, $|H_0| = p$ with p a prime. We also observe that any proper subgroup of $\langle a \rangle$ must coincide with H_0 because it is not isolated. This leads to $|\langle a \rangle| = p^2$.

Suppose that G possesses a subgroup H of order p^3 containing $\langle a \rangle$. Then H is of one of the following types: $\mathbb{Z}_{p^3}$, $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$, the non-abelian group of order p^3 and exponent p^2, D_8 or Q_8. It is easy to see that if $H \cong \mathbb{Z}_{p^3}$ then $\langle a \rangle$ is not isolated, while if $H \cong D_8$, then the non-cyclic subgroups of order 4 of H are not isolated, contrary to our assumption. In the other cases, there exists $\langle b \rangle \leq H$ with $|\langle b \rangle| = p^2$ and $\langle b \rangle \cap \langle a \rangle = H_0$, proving that $\langle a \rangle$ is again not isolated, a contradiction. Thus, $\langle a \rangle$ is a Sylow p-subgroup of G.

Suppose now that $|G|$ has a prime divisor $q \neq p$, and let $K \leq G$ with $|K| = q$. Then, $H_0 K \leq G$, $a \notin H_0 K$ and

$$\langle a \rangle \cap H_0 K = H_0 \neq 1,$$

showing that $H_0 K$ is not isolated in G, a contradiction.

Hence $G = \langle a \rangle \cong \mathbb{Z}_{p^2}$, as desired.

(c) Let Isolated$(G) = L(G) \setminus \{H_1, H_2\}$. First of all, we assume that H_1 and H_2 are normal in G. Let $a \in G \setminus H_1$ such that $\langle a \rangle \cap H_1 \neq 1$. Then $\langle a \rangle \cap H_1$ is not isolated, implying that $\langle a \rangle \cap H_1 \in \{H_1, H_2\}$. We distinguish the following two cases:

Case 1. $\langle a\rangle \cap H_1 = H_1$.

Then $H_1 \subset \langle a\rangle$. Since proper nontrivial subgroups of $\langle a\rangle$ are not isolated, we infer that $\langle a\rangle$ has either exactly one proper nontrivial subgroup, namely H_1, or exactly two, namely H_1 and H_2. So, we have two possibilities:

Subcase 1.1. H_1 is the unique proper subgroup of $\langle a\rangle$.

Then, $|\langle a\rangle| = p^2$ and $|H_1| = p$, where p is a prime.

Suppose first that H_1 is the unique minimal subgroup of G. This implies that G is either a cyclic p-group or a generalized quaternion 2-group. Then it is easy to see that G has exactly two non-isolated subgroups if and only if $G \cong \mathbb{Z}_{p^3}$.

Suppose now that G possesses minimal subgroups different from H_1, and let K be such a subgroup. Then, $a \notin H_1K$ and

$$\langle a\rangle \cap H_1K = H_1 \neq 1,$$

that is H_1K is not isolated in G. Since $H_1K \neq H_1$, we get $H_1K = H_2$. Thus H_2 contains all minimal subgroups of G.

Observe also that $\langle a\rangle$ is normal in G. Indeed, for every $x \in G$, we must have $\langle (a^x)^p\rangle = H_1 = \langle a^p\rangle$ because $\langle (a^x)^p\rangle$ is not isolated. So, $\langle a\rangle \cap \langle a^x\rangle \neq 1$, and since $\langle a\rangle$ is isolated, we conclude $\langle a\rangle = \langle a^x\rangle$.

Let $H \neq \langle a\rangle, H_2$ be a maximal subgroup of $\langle a\rangle H_2$. Then H is isolated in G, and therefore $\langle a\rangle \cap H = 1$. Since $\langle a\rangle H = \langle a\rangle H_2$, it follows that H is a minimal subgroup of G, which leads to $H \subset H_2$. Thus $\langle a\rangle H_2$ has exactly two maximal subgroups, namely $\langle a\rangle$ and H_2, and consequently, it is cyclic. This implies that $\langle a\rangle$ is not isolated in G, a contradiction.

Subcase 1.2. H_1 and H_2 are the unique proper subgroups of $\langle a\rangle$.

Then we have either

$$|\langle a\rangle| = p^3 \text{ and } |H_1| = p,\ |H_2| = p^2, \quad \text{where } p \text{ is a prime}$$

or

$$|\langle a\rangle| = pq \text{ and } |H_1| = p,\ |H_2| = q, \quad \text{where } p, q \text{ are distinct primes.}$$

In both cases, if G would have a minimal subgroup K different from H_1 and H_1, H_2, respectively, then $a \notin H_1K$ and

$$\langle a\rangle \cap H_1K = H_1 \neq 1,$$

i.e., H_1K is non-isolated in G. Since obviously $H_1K \neq H_1, H_2$, this contradicts our hypothesis. Thus H_1 and H_1, H_2, respectively, are the unique minimal subgroups of G, proving that G is one of the following groups:

- $\mathbb{Z}_{p^m}$, where p is a prime and $m \in \mathbb{N}$;
- Q_{2^n}, where $n \in \mathbb{N}$, $n \geq 3$;
- $\mathbb{Z}_{p^m q^n}$, where p, q are distinct primes and $m, n \in \mathbb{N}$;
- $\mathbb{Z}_{p^m} \times Q_{2^n}$, where p is an odd prime and $m, n \in \mathbb{N}$, $n \geq 3$.

In all cases, one can easily check that G has exactly two non-isolated subgroups if and only if it coincides with $\langle a \rangle$, as desired.

Case 2. $\langle a \rangle \cap H_1 = H_2$.

Then, $H_2 \subseteq \langle a \rangle$ and $H_2 \subset H_1$. If $H_2 = \langle a \rangle$, then we get $\langle a \rangle \subseteq H_1$, a contradiction. Thus $H_2 \neq \langle a \rangle$. Again, proper subgroups of $\langle a \rangle$ are not isolated in G. If H_2 is the unique proper subgroup of $\langle a \rangle$, then we proceed similarly with Subcase 1.1, while if H_1 and H_2 are the unique proper subgroups of $\langle a \rangle$, then we proceed similarly with Subcase 1.2.

Assume now that H_1 is not normal in G. Since conjugates of non-isolated subgroups are non-isolated, it follows that H_1 and H_2 are conjugate, and every element of G either normalizes both subgroups, or exchanges them. Therefore, $N = N_G(H_1) = N_G(H_2)$ contains both H_1 and H_2, so in particular, N is isolated in G.

Clearly, any subgroup of N that is isolated in G is also isolated in N. Let $a \in G \setminus H_i$ such that $\langle a \rangle \cap H_i \neq 1$. Then, $\langle a \rangle \cap N \neq 1$, and since N is isolated, it follows that $a \in N$. Thus, H_i is also not isolated in N. We conclude that $|\text{Isolated}(N)| = |L(N)| - 2$, and the two non-isolated subgroups of N are normal in N.

But then by the discussion above, with N playing the role of G, we know that H_1 and H_2 have different orders, contradicting that they are conjugate in G. This completes the proof. $\square$

ACKNOWLEDGMENTS. The author is grateful to the reviewers for their remarks which improved the previous version of the paper.

References

[1] V. M. BUSARKIN, The structure of isolated subgroups in finite groups, *Algebra i Logika Sem.* **4** (1965), 33–50.

[2] V. M. BUSARKIN, Groups containing isolated subgroups, *Sibirsk. Mat. Ž.* **9** (1968), 752–756.

[3] K. N. CHENG, M. DEACONESCU, M.-L. LANG and W. J. SHI, Corrigendum and addendum to: "Classification of finite groups with all elements of prime order" by Deaconescu, *Proc. Amer. Math. Soc.* **117** (1993), 1205–1207.

[4] Encyclopedia of Mathematics, `https://encyclopediaofmath.org/wiki/Isolated_subgroup`.

[5] M. Deaconescu, Classification of finite groups with all elements of prime order, *Proc. Amer. Math. Soc.* **106** (1989), 625-–629.

[6] Z. Janko, Finite p-groups with some isolated subgroups, *J. Algebra* **465** (2016), 41–61.

[7] A. G. Kurosh, The Theory of Groups, Second English Edition, two volumes, *Chelsea Publishing Co.*, *New York*, 1960.

[8] R. Schmidt, Subgroup Lattices of Groups, De Gruyter Expositions in Mathematics, Vol **14**, *Walter de Gruyter* & *Co.*, *Berlin*, 1994.

[9] Q. Song and L. An, Isolated subgroups of finite p-groups, *Bull. Aust. Math. Soc.* **105** (2022), 449–457.

[10] M. Suzuki, Group Theory, I, II, *Springer-Verlag*, *Berlin – New York*, 1982, 1986.

[11] M. Tărnăuceanu, Isolated subgroups of finite abelian groups, *Czechoslovak Math. J.* **72(147)** (2022), 615–620.

MARIUS TĂRNĂUCEANU
FACULTY OF MATHEMATICS
"AL.I. CUZA" UNIVERSITY
BD. CAROL I, NO. 11
IAŞI - 700506
ROMANIA

E-mail: tarnauc@uaic.ro
URL: http://www.math.uaic.ro/martar

(Received December 1, 2021; revised July 19, 2022)